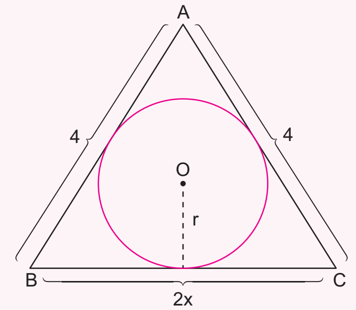
APPLICATIONS OF DERIVATIVE
MINIMUM-MAXIMUM PROBLEMS

Problem

Let ABC be a triangle with sides |AB| = |AC| = 4 and the radius of ABC's incircle is r.

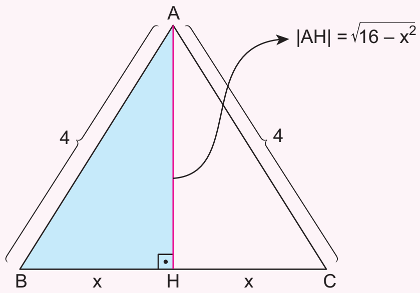
Find |BC| for the maximum value of r.

Solution



$2u = 4 + 4 + 2x \Rightarrow u = 4 + x$

$[\widehat{ABC}] = u \cdot r = (4 + x) \cdot r$ .... (i)



$[\widehat{ABC}] = \frac{1}{2} \cdot |BC| \cdot |AH| = x \cdot \sqrt{16 - x^2}$ .... (ii)

From (i) and (ii),

$$(4+x) \cdot r = x \cdot \sqrt{16 - x^2} \Rightarrow r = \frac{x \cdot \sqrt{16 - x^2}}{4 + x} = \frac{\sqrt{16x^2 - x^4}}{4 + x} = \sqrt{\frac{16x^2 - x^4}{x^2 + 8x + 16}}$$

$$f(x) = \frac{16x^2 - x^4}{x^2 + 8x + 16} \Rightarrow f'(x) = \frac{(32x - 4x^3) \cdot (x^2 + 8x + 16) - (2x + 8) \cdot (16x^2 - x^4)}{(x^2 + 8x + 16)^2}$$

$$f'(x) = 0 \Rightarrow (32x - 4x^3) \cdot (x+4)^{\not 2} = 2 \cdot \cancel{(x+4)} \cdot (16x^2 - x^4)$$

$$\Rightarrow 4\cancel{x} \cdot (8 - x^2) \cdot \cancel{(x+4)} = 2.x^{\not 2} \cdot \underbrace{(16 - x^2)}_{(4-x)\cancel{(4+x)}}$$

$$\Rightarrow \overset{2}{\cancel{4}} \cdot (8 - x^2) = \cancel{2}x \cdot (4 - x)$$

$$\Rightarrow 16 - 2x^2 = 4x - x^2 \Rightarrow x^2 + 4x - 16 = 0$$

$$\Rightarrow (x + 2)^2 = 20$$

$$\Rightarrow x + 2 = 2\sqrt{5}$$

$$\Rightarrow x = 2\sqrt{5} - 2$$

Finally, $|BC| = 2x = 4\sqrt{5} - 4$.

While r is maximum, then $\cos(\widehat{C}) = \frac{1}{\varphi}$

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Alpaslan CERAN